An Investigation of Representations of Combinatorial Auctions

(Extended Abstract)

David Loker and Kate Larson
University of Waterloo
200 University Ave. W
Waterloo, ON, Canada N2L 3G1
{dloker, klarson}@cs.uwaterloo.ca

ABSTRACT

Combinatorial auctions (CAs) are an important mechanism for allocating multiple goods while allowing self-interested agents to specify preferences over bundles of items. Winner determination for a CA is known to be NP-complete. However, restricting the problem can allow us to solve winner determination in polynomial time. These restrictions sometimes apply to the CA's representation. There are two commonly studied, and structurally different graph representations of a CA: bid graphs and item graphs. We study the relationship between these two representations.

We show that for a given combinatorial auction, if a graph with maximum cycle length three is a valid item graph for the auction, then its bid graph representation is a chordal graph. Next, we present a new technique for constructing item graphs using a novel definition of equivalence among combinatorial auctions. The solution to the WDP for a given CA can easily be translated to a solution on an equivalent CA. We use our technique to simplify item graphs, and show that if a CA's bid graph is chordal, then there exists an equivalent CA with a valid item graph of treewidth one, for which a solution to the WDP is known to be efficient. This result demonstrates how CA equivalence can simplify the structure of item graphs and lead to more efficient solutions to the WDP, which are also a solutions to the WDP for the original auctions.

Categories and Subject Descriptors

F.2 [**Analysis of Algorithms and Problem Complexity**]: Miscellaneous—*combinatorial auctions, winner determination problem*

General Terms

Algorithms and Theory

Keywords

Auction and mechanism design, combinatorial auctions, parameterized complexity

Cite as: An Investigation of Representations of Combinatorial Auctions (Extended Abstract), David Loker and Kate Larson, *Proc. of 9th Int. Conf. on Autonomous Agents and Multiagent Systems (AAMAS 2010)*, van der Hoek, Kaminka, Lespérance, Luck and Sen (eds.), May, 10–14, 2010, Toronto, Canada, pp. 1481-1482

1. PRELIMINARIES

Combinatorial Auctions

A combinatorial auction consists of a set of n agents, and m items to be auctioned. The set of agents will be denoted by $N = \{1, 2, \ldots, n\}$ and the set of items by M. For any subset $S \subseteq M$, any agent i can place a bid $b_i(S) \in \mathbb{Z}$ on S. We assume that $b_i(S) \geq 0$ for all $S \subseteq M$ and that the agents are self-interested.

An atomic bid, denoted (S, p), includes a set of items $S \subseteq M$ and its bid value, $p \geq 0$. We assume that for each agent i, bids are submitted as a set of atomic bids, $\{(S_{i1}, p_{i1}), \ldots, (S_{ir_i}, p_{ir_i})\}$.

The winner determination problem, which calculates the allocation of goods to agents can be done by solving an integer program, which is NP-hard. We refer the reader to [2] for a more thorough introduction to CAs and the WDP.

Bid Graphs and Item Graphs

Each vertex in a bid graph is a bid, and an edge exists between two vertices if the two bids they represent share an item. By definition, a bid graph is the intersection graph of the distinct atomic bids.

Item graphs are a representation of CAs altogether different from bid graphs. We will be interested in restricting our attention to CAs that can be represented by an item graph with a specific structure, in order to gain insight into the structure of the auction's bid graph. Informally, in a valid item graph of the CA, the bids must be connected induced subgraphs of the item graph. Formally, we have the following definition:

Definition 1.0.1. *Given a CA, a* valid item graph $G = (M, E)$ *representing the given CA must satisfy the following conditions: each item is represented by exactly one vertex in the graph, and for each atomic bid* (S, p)*, the induced subgraph of* G *on the vertices contained in* $S \subseteq I$ *must be a connected graph.*

An item graph can represent many different CAs. In contrast, one may translate a bid graph to and from a CA and in polynomial time. Any CA obtained from a bid graph yields the same solution to WDP, ignoring the case where multiple agents submitted the same highest bid on a bundle of items. Since an item graph allows for many different combinations of bids and bid values, the solution to WDP could be different for each CA that the item graph may represent. Obviously then, we cannot formulate a specific bid graph given an item graph. However, there are relationships between the structures of both representations. Our goal is to explore these relationships.

2. RESULTS

While item graphs and bid graphs differ in how they represent CAs, the underlying instance of the auction they represent is the same. We wish to determine how bid graphs and item graphs relate to one another. By better understanding the relationships between item graphs and bid graphs, we may be able to create more efficient algorithms for solving WDP and, as we will see, find a new way of looking at CAs in general.

First, we consider CAs that have valid item graphs with a maximum cycle length.

THEOREM 2.0.2. *For a given CA, if there exists a valid item graph $G_I = (V_I, E_I)$ representing the auction such that the maximum length of any cycle in G_I is three, then the bid graph for the auction is chordal.*

The result does not extend to cycles of length four or item graphs of treewidth two. Further, the converse does not hold. This impasse lead to our next avenue of investigation: modifying combinatorial auctions.

The motivation behind studying the modification of CAs begins with the fact that item graphs are hard to construct. As shown by Conitzer *et al.*, constructing a valid item graph with the fewest edges is NP-complete [1]. Given a CA, Conitzer *et al.* construct a valid item graph of treewidth one in polynomial time, if the graph exists [1]. However, as shown by Gottlob and Greco, it is NP-hard to decide whether or not a combinatorial auction has a valid item graph of treewidth three [4]. On the other hand, bid graphs can always be constructed in polynomial time.

Throughout the literature, it is assumed that a CA is fixed before the construction of a valid item graph. We take a novel perspective by modifying the CA in order to achieve a valid item graph of smaller treewidth, while maintaining the same solution to WDP. We do this by introducing a notion of CA equivalence.

Intuitively, two bids graphs are equivalent if they are isomorphic, and two CAs are equivalent if their bid graphs are equivalent. With this definition of equivalence, we obtain the following result:

THEOREM 2.0.3. *Let CA denote a combinatorial auction that has a valid item graph $G_I = (V_I, E_I)$. If the bid graph G_B of CA is chordal, then there exists a combinatorial auction CA' equivalent to CA that has a tree G'_I as a valid item graph.*

The item graph for the equivalent CA can be found in polynomial time, given the bid graph of the original auction [3]. With our notion of CA equivalence, if the bid graph is chordal, then even if the smallest treewidth for the item graph is arbitrarily larger than one, there still exists an equivalent auction with a tree as a valid item graph. This perspective on CA equivalence and the existence of item graphs of small treewidth has never been shown before.

With this result, it becomes difficult to gauge the usefulness of item graphs. If it is possible for a CA to have a valid item graph of treewidth one while another equivalent CA does not, then it is possible that we are translating our auction to a less efficient form. Further, there may or may not exist a CA with a chordal bid graph for which all valid item graphs have treewidth at least tw, for some large tw. That is, it is unclear how big of an improvement is possible using the equivalent CA technique. Given this potentially large lack of consistency between item graph representations of equivalent CAs, the construction of item graphs, as previously described in the literature, appears to be flawed.

3. CONCLUSIONS

We present a new technique for constructing item graphs using our new notion of CA equivalence. Currently the item graph construction process involves finding an item graph for a *fixed* CA. We demonstrate that this is not necessarily optimal.

Since it is NP-hard to find a valid item graph of treewidth $tw \geq 3$ [4], it may be better to find equivalent CAs for which finding a valid item graph of treewidth tw is easy. This brings to question the practicality of studying item graphs of bounded treewidth. The new construction technique that we introduce opens another avenue of investigation for determining the practicality of item graphs of bounded treewidth. We initiate the study of combinatorial auction modification. For future work, it may be interesting to consider alternative constructions.

It would also be interesting to investigate relationships between bid graphs and item graphs of treewidth greater than one. Our new notion of CA equivalence and the resulting construction technique may be useful in this investigation. Further, Gottlob and Greco recently introduced a new method for qualifying hypergraphs of CAs, which they refer to as hypertrees of bounded hypertree width [4]. Item graphs of bounded treewidth are a special case of hypertrees of bounded hypertree width, and as such it would be interesting to see the parallels, if any, between their relationships with bid graphs. Despite the fact that item graphs of bounded treewidth are a special case of this new model, we can construct a hypertree of bounded hypertree width for a given CA, should one exist, in time that is polynomial in the size of the auction [4]. Does our new construction method account for why an item graph of bounded treewidth was previously NP-hard to construct, for treewidth larger than two, while a hypertree of bounded hypertree width is polynomial to construct for any fixed hypertree width?

4. ACKNOWLEDGMENTS

The authors would like to thank Naomi Nishimura for her help with this work.

5. REFERENCES

[1] V. Conitzer, J. Derryberry, and T. Sandholm. Combinatorial auctions with structured item graphs. In *AAAI*, pages 212–218, 2004.

[2] S. de Vries and R. Vohra. Combinatorial auctions: A survey. *Informs Journal on Computing*, 15(3):284–309, 2003.

[3] F. Gavril. The intersection graphs of subtrees in trees are exactly the chordal graphs. *J. of Comb. Theory, Series B*, 16:47–56, 1974.

[4] G. Gottlob and G. Greco. On the complexity of combinatorial auctions: Structured item graphs and hypertree decompositions. In *ACM Conference on Electronic Commerce*, pages 152–161, 2007.